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Geometry Teaching in Transition: An Investigation on the Importance of School Geometry in Primary Education

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Mathematics instruction in primary school has been influenced by many policy changes and empirical findings in the previous two decades. Geometry lessons, in particular, were re-evaluated due to a paradigm change and, consequently, were attributed a new meaning within the mathematics curriculum worldwide. The present paper focuses on this paradigm shift in the sense of the evaluation to what extent both the didactical potential and the practical value of geometry instruction in elementary education are currently recognised and utilised by primary grade teachers. In total, 120 primary grade teachers participated in the study. The results showed that there had been positive recognition of the didactical potential of school geometry by the teachers over the previous two decades; however, it lacked actual implementation in school practice for different reasons. The results are discussed not only with regard to the latter of these but also with regard to their theoretical and practical implications.

Keywords: geometry, geometry instruction, primary education, in-service teachers

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Poučevanje geometrije v tranziciji: poizvedba pomembnosti šolske geometrije v osnovni šoli

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Poučevanje matematike v osnovni šoli je bilo v zadnjih dveh desetletjih pod vplivom številnih preoblikovanj politik pa tudi empiričnih ugotovitev. Pouk geometrije je bil ponovno ovrednoten skladno s paradigmatsko spremembo, čemur je sledilo pripisovanje novega pomena geometriji znotraj učnih načrtov za matematiko po svetu. Prispevek se osredinja na ta paradigmatski premik v smislu evalvacije, do katere mere sta didaktični potencial in praktična vrednost poučevanja geometrije v osnovni šoli trenutno priznana in izkoriščena pri učiteljih. Skupno je v raziskavi sodelovalo 120 učiteljev razrednega pouka. Rezultati kažejo, da lahko zaznavamo pozitivno prepoznavo didaktičnega potenciala šolske geometrije pri učiteljih v zadnjih dveh desetletjih, vendar pa do dejanske izvedbe v šolsko prakso zaradi različnih razlogov ni prišlo. Rezultati niso pojasnjeni samo z upoštevajočim primanjkljajem, ampak sklicujoči se na teoretične in praktične posledice.

Ključne besede: geometrija, poučevanje geometrije, osnovnošolsko izobraževanje, učitelji

Introduction

When Winter published an article titled 'What's the purpose of geometry in primary education?' in 1976, the debate about the relevance of geometry for students in the early grades was already in full swing. Even though there is a clear consensus that geometry instruction is indispensable even at the elementary school level (Franke & Reinhold, 2016; Sinclair & Bruce, 2015; Sitter, 2019), the fundamental debate has not been completely settled (Eichler, 2005). Arguably, there is little reason to do so, as the status of elementary school geometry instruction is still considered unsatisfactory in many cases, especially compared to that of arithmetic (Backe-Neuwald, 2000; Sinclair & Bruce, 2015). Furthermore, 'study after study shows that students perform quite poorly on a wide range of geometry tasks' (Sinclair & Bruce, 2015, p. 319) independent of the country, which calls for emergent attention to primary school geometry (e.g., Glasnović Gracin & Kuzle, 2018).

Backe-Neuwald's (2000) holistic study 'Meaningful geometry in primary school' provided the first impressions of various aspects of geometry teaching in German elementary schools two decades ago. However, the picture emerging from the study was full of contradictions. Despite manifold advantages that lift it in a positive way from the rest of mathematics, geometry teaching appeared as the stepchild of mathematics in the eyes of the respondents (Backe-Neuwald, 2000). The main reason for this was the uncertainty of the teachers, both regarding the selection of central geometry contents and the criteria that determined their significance.

In the previous two decades, geometry lessons were re-evaluated due to a paradigm change and were assigned new meaning within school mathematics in primary and secondary education (Franke & Reinhold, 2016; Kultusministerkonferenz [KMK], 2005; Mammana & Villani, 1998). Only a few studies (e.g., Sitter, 2019; Wiese, 2016) have been based on recent data following the paradigm shift. This paper focuses on recent developments and discussions, with a particular interest in teachers and their attitudes toward teaching geometry, with the aim of providing additional empirical evidence based on current data on the state of elementary school geometry. Concretely, the focus of the investigation was the extent to which both the didactical potential and the practical value of geometry instruction in elementary education are recognised and utilised by primary grade teachers nowadays. Understanding the status quo of current geometry teaching is the basis for different policy decisions, such as the further development of mathematics curricula and teacher training, and, therefore, highly relevant for the improvement of geometry teaching in primary education.

Theoretical Background

In this section, I first present the reasons for teaching geometry from early grades onward, which is followed by empirical research on the topic. The section ends with the two research questions that guided the study.

Reasons for Teaching Geometry in Primary Education

The goals and aspects of the modern didactics of geometry have changed and evolved (see Franke & Reinhold, 2016; Radatz & Rickmeyer, 1991; Winter, 1976; Wittmann, 1999). This didactic potential of geometry can be combined with other mathematical topics and has great potential to complement them and make them comprehensible through other approaches (Franke & Reinhold, 2016). In the following, the topics and contents relevant to modern geometry didactics are outlined. The detailed descriptions of the goals and contents of good geometry teaching are intended to illustrate their relevance for mathematics teaching as a whole and highlight the versatile didactical potential of geometry for primary education.

The proponents of an early introduction of geometric content in school assert various reasons. First, geometry promotes knowledge and skills that have relevance and overlap with a wide range of school content, not only in mathematics. For mathematics, a discipline in which, more than in many other subjects, the contents build on each other (van den Heuvel-Panhuizen, 2008), this connection seems to be obvious, and in that manner provides not only a foundation for further geometry instruction (Franke & Reinhold, 2016) but also for other mathematical topics (Hattermann et al., 2015). Furthermore, Bauersfeld (1992) argued that geometry is the basis of arithmetic, both in terms of content and teaching-learning processes in general; he called this the 'genetic connection' (p. 7) between geometry and arithmetic whose argumentation was based on a constructivist view, namely by putting thought processes over the correctness of the final results or imparting competencies instead of poorly contextualised subject knowledge. Consequently, geometry can contribute to absorbing the strong heterogeneity of learning preconditions and socialisations present in elementary school and to creating sustainable learning foundations from it (Krauthausen, 2018; Wollring, 2007). These specific features of geometry lessons (e.g., possibilities for activity-based teaching, discovery learning, action-oriented instruction) relate more strongly to the students (Radatz & Rickemeyer, 1991) and, in that manner, may influence students' motivation and help them develop a positive attitude towards mathematics (Krauthausen, 2018). In addition to content areas of mathematics, geometry instruction also facilitates access to process-related competencies, such as problem solving and argumentation (KMK, 2005; NCTM, 2000). Researchers (Kuzle & Bruder, 2016; Wittmann, 1999) emphasised that geometry instruction can contribute to the development of problem-solving abilities due to the richness of geometric content, which opens possibilities for the heuristic approach in the sense of a discovering, trying out, and composing and decomposing procedures.

Geometry is viewed in a special way as a tool for acquiring skills, not only, as mentioned above, in the area of school competencies, but much more generally, such as the acquisition of intellectual, cognitive and practical life competencies (Franke & Reinhold, 2016; Graumann, 1994; van den Heuvel-Panhuizen, 2008; Wittmann, 1999). For example, language is permeated by mathematics concepts in the sense of knowledge of the mathematical necessity of a particular mathematical relationship (Simon, 2017). In everyday speech, we use geometric concepts, such as positional relations (Franke & Reinhold, 2016; van den Heuvel-Panhuizen, 2008). Three-dimensional space in particular is difficult for young students to access in planar representations (Eichler, 2005). Geometry can bridge this gap by helping students to gently navigate this process of abstraction with constant recourse to their familiar environment, and in that manner may support the development of the orientation ability and the ability to operate with objects mentally (Franke & Reinhold, 2016; van den Heuvel-Panhuizen, 2008). The process is complete when children learn to comprehend their environment and to see it in a different light through geometry instruction (Eichler, 2005; Franke & Reinhold, 2016). Winter speaks of the 'spatial reality of the child being carefully disciplined' (Winter, 1976, p. 14) when natural experiences are reflected on, analysed, and structured in the classroom. Nowadays, this list certainly needs to be supplemented by digital technologies, which can further support the teaching and learning of geometry (Jones, 2000; Sinclair & Bruce, 2015).

Empirical Studies on the Role of Geometry Teaching

The much-cited work of Backe-Neuwald (2000) still represents a milestone in the contemporary empirical examination of primary school geometry and its role in the school structure. It reflected the importance of geometry in the teaching practice at the end of the 1990s from the perspective of 108 in-service and 22 trainee teachers using a comprehensive questionnaire. The results of Backe-Neuwald's work have since been followed up twice in the form of a replication (Sitter, 2019) or extension study (Wiese, 2016).

The teachers' image of geometry lessons was rather ambivalent (Backe-Neuwald, 2000). When the teachers were asked about associations with elementary school geometry lessons, some described it as 'a welcome change' or 'an exciting thing', while others considered it 'secondary' or 'not important' (Backe-Neuwald, 2000, pp. 16–18). Furthermore, many teachers suggested that the short teaching time should be filled with more important mathematical content, especially arithmetic. Nevertheless, Backe-Neuwald (2000) reported that geometry instruction was described in a positive manner compared to arithmetic instruction, with the teachers emphasising teaching principles more typical for geometry instruction, such as action-orientation, discovery-based learning, problem-oriented teaching, orientation to the children's real lives, and working with hands-on materials and manipulatives. Regarding the latter, Backe-Neuwald (2000), however, reported that many teachers shied away from the preparatory intensity of geometry lessons as was also reported in a later study by Sitter (2019). Furthermore, the teachers stated that geometry lessons were increasingly characterised by partner and group work, allowed free work and open lessons, and could be taught across subjects. The majority of the teachers surveyed felt that geometry instruction had a motivating effect on the students.

When it came to the advantages of teaching geometry, the three benefits that were by far most often chosen by the teachers were 'offers children many opportunities to make independent discoveries', 'promotes and supports spatial visualisation ability', and 'makes an important contribution to the development of reality', which are aligned with the literature in geometry didactics (e.g., Franke & Reinhold, 2016; Sinclair & Bruce, 2015; Winter, 1976). Furthermore, the statement 'lays the foundation for later systematic geometry teaching in secondary school' was also recognised by the teachers. This may indicate a corresponding long-term perspective of the teachers or be the reason for the neglect in elementary school, since teachers may classify geometry instruction as a topic of secondary school (Backe-Neuwald, 2000). Despite the teachers generally showing positive attitudes toward geometry instruction, they also provided several reasons for neglecting it (Backe-Neuwald, 2000). Geometry was predominantly taught in Grades 1–2 and less so in Grades 3–4. Backe-Neuwald (2000) assumed different reasons for this result. Firstly, it may be that the teachers felt pressured to cover all arithmetic topics which prevented them from teaching geometry in Grades 3–4. Secondly, it may be that the teachers did not feel competent in teaching the subject (Backe-Neuwald, 2000). The latter was also supported by Hofbauer (2018) who reported that geometry had a low priority for many secondary in-service teachers in their educational studies and, therefore, felt inadequately trained in this regard. Wiese's (2016) comparative study with 16 primary grade in-service teachers confirmed the earlier results of Backe-Neuwald (2000). On average, geometry instruction took up about 8% of all mathematics lessons in the school year, however, the values fluctuated between 2% and 19% among the individual teachers (Wiese, 2016). Wiese (2016) put this in relation to the roughly calculated weight of the geometry in the curriculum, which she specified as 20% to 30% (Wiese, 2016). Accordingly, she concluded that geometry lessons were still not given the weight that the curriculum assigned them.

Overall, Backe-Neuwald (2000) found that many of the respondents dealt extensively with geometry instruction on a theoretical level, and some showed a very reflective and self-critical attitude toward their own geometry instruction. However, the teachers' answers reflected the disparity between the importance they attached to the subject and the significance that the subject had in their teaching practice. In total, 80% of the teachers surveyed agreed with the thesis that geometry instruction was neglected in elementary school, despite the stated advantages. Thus, there was a clear gap between the aspirations and the reality of geometry instruction in elementary education. Consequently, Backe-Neuwald (2000) concluded that the mathematics curriculum should be reconsidered, namely geometric and arithmetic content should be more closely interlinked and prepared in the sense of modern mathematics didactics. Since Backe-Neuwald's findings, however, the mathematics curriculum, and with it also geometry instruction, have been subject to significant external changes (Mammana & Villani, 1998). The impulses are due to, on the one hand, the results of the large-scale studies (i.e., PISA, TIMSS), and the revaluation of geometry in the context of the revised (inter-)national standards, on the other; this makes a reassessment of existing empirical findings necessary.

Research Questions

Based on the above theoretical considerations and empirical results, the following research questions were investigated:

1. What didactical potential do in-service teachers assign to elementary school geometry?
   - To what extent does the didactical potential of different geometry teaching goals and aspects discussed in the modern geometry didactics correspond to current teaching practices?
   - Considering the didactical potential of geometry teaching, which goals and aspects are relevant to in-service teachers?

2. What practical value do in-service teachers assign to elementary school geometry?

Method

Participants

For this study, a mixed-methods research design was chosen using a convenience sample. Here, elementary schools were selected through existing contacts with the researcher's university. Of the 159 schools contacted, only 45 participated in the study; of the 176 questionnaires distributed, 120 of them were returned anonymously. The sample of 120 in-service primary teachers (Grades 1–6) consisted of 22 male (18.3%) and 97 female teachers (80.8%). One person did not provide gender information. A total of 90 teachers taught mathematics as subject specialists (75%) and 29 of them as non-subject specialists (24.2%). The data of one person was not provided. In terms of professional experience, the following picture emerged: 16 teachers (13.3%) have been teaching mathematics for less than or up to two years, 18 (15%) for up to 5 years, 19 (15.8%) for up to 10 years, 16 (13.3%) for up to 20 years, and 51 (42.5%) for more than 20 years.

Data Collection Instrument

The main source of data was a questionnaire on the state of the art of school geometry in primary grades that was based on an adaptation of the instrument from the work of Backe-Neuwald (2000). Additionally, new items or statements for a specific item were developed on the basis of literature published in the previous 20 years that (amongst other factors) reflected the paradigm shift, curriculum developments, and factors influencing geometry instruction (Franke & Reinhold, 2016; Krauthausen, 2018; Senatsverwaltung für Bildung, Jugend und Wissenschaft Berlin, Ministerium für Bildung, Jugend und Sport des Landes Brandenburg [RLP], 2015). To cover a wide field of research and different research questions, the items about geometry teaching were very broad. The questionnaire consisted of six sections: (1) personal information (e.g., gender, teaching experience, professional background), (2) characteristics of teaching geometry (i.e., associations regarding teaching geometry, number of lessons dealing with geometric topics per grade level, instruction form, teaching principles, topics covered, use of digital tools, teaching sources), (3) material (i.e., importance of material in geometry teaching, goals of using material for teaching purposes, concrete material being used regarding teaching a specific geometry topic), (4) goals and aspects of teaching geometry (i.e., importance of teaching of geometry for its application in everyday life, characterisation of students in geometry classes in comparison with other mathematics areas (e.g., motivation, interest, concentration), advantages of teaching geometry), (5) neglect of teaching geometry (i.e., teachers' opinion on whether geometry is neglected in school mathematics, evaluation of the reasons leading to the neglect of geometry instruction), and (6) personal attitude toward teaching geometry (i.e., emotions associated with teaching geometry).

Each section consisted of items with both open and closed questions. Thus, the questionnaire was self-contained and formed a coherent instrument in its structure and design, which should have always been filled out completely, even if (as in the present work) a selection of the given answers was made afterwards along with the own main points of investigation. The former enabled comparability between the groups studied, whereas the latter allowed the participants the opportunity to write down their own opinions without imposing the researcher's view. In this paper, I focus on sections (2) and (4) of the questionnaire, namely 'characteristics of teaching geometry', and 'goals and aspects of teaching geometry', respectively. With respect to the former, three items were analysed and with respect to the latter one item was analysed.

Data Analysis

The questionnaire was for the most part distributed on the site after agreement with the school management; in isolated cases, it was also sent by email and returned to the author within one week. The questionnaires were analysed after all the data had been collected.

To determine what didactical potential in-service teachers assign to elementary school geometry two items were analysed: an open-ended item from section (2) 'characteristics of teaching geometry' and a standardised item from section (4) 'goals and aspects of teaching geometry'. The open-ended item was as follows: 'Geometry teaching in elementary school is for me…' which allowed an insight into teachers' goals and aspects of their geometry teaching. The analysis of the item was based on qualitative content analysis according to Mayring (2000). Here, the theory-based deductive category system was used, which resulted from the literature review presented earlier in the paper and was applied to the teachers' answers. In the qualitative analysis step, the deductively derived categories were methodically assigned to text passages. The core element here is the precise definition of the given categories (Mayring, 2000). In this study, the categories emerged from the literature research and the current state of the modern geometry didactics, providing the foundation for a coding manual for the evaluation of the qualitative item. Accordingly, the material was coded based on the following eight categories:

- Motivation: geometry instruction has a motivating effect on students through alternative instructional concepts and a sense of achievement by experiencing success (Krauthausen, 2018).
- Sustainable learning environments: in geometry instruction, cooperative, differentiable, and action-oriented learning environments have a high didactical potential (Wollring, 2007).
- Problem solving: geometry lessons offer many opportunities to support the development of problem-solving abilities (Kuzle & Bruder, 2016).
- Opening up reality: geometry aids in understanding reality, and trains learners in everyday practical competencies (Graumann, 2009; van den Heuvel-Panhuizen, 2008).
- Spatial visualisation ability: geometry instruction supports the development of the ability to orientate oneself in three-dimensional space, and to operate mentally (Franke & Reinhold, 2016; Sinclair & Bruce, 2015; van den Heuvel-Panhuizen, 2008).
- Acquisition of arithmetic concepts: geometry supports and complements the acquisition of arithmetic concepts (Bauersfeld, 1992; Franke & Reinhold, 2016; van den Heuvel-Panhuizen, 2008).
- Geometric concept formation: the knowledge of geometric concepts provides a foundation for further geometry instruction and supports the development of knowledge to systematise geometric concepts (Franke & Reinhold, 2016).
- Basic knowledge: geometry (content) is part of general knowledge and provides a foundation for understanding other mathematical and historical topics (Hattermann et al., 2015).

Concretely, the text material was examined to determine to what extent the categories can be applied and whether there are problems of demarcation between the categories. In the case of the present work, the category system could be tested as suitable for all texts. However, there were statements that could not be categorised in the presented category system as they contained no content pertaining to the didactical potential of geometry teaching. With respect to the former, after the first pass with the help of the interpretation rules, which are based on Mayring's work (2000) regarding the applied techniques of 'paraphrasing' and 'generalisation', eight categories were formed. All collected statements of the teachers were shortened in paraphrasing to the essential content of the statement and then by generalisation, the main statement was extracted. These main statements were assigned to each category. The resulting category system now contained those aspects that could be filtered out and summarised from the available text material in a theory-based manner on the basis of the defined characteristics for 'goals and aspects of geometry instruction'. With respect to the statements that did not fit the developed category system, an additional category system needed to be developed as they contained no content mathematics-related statements but more affect-related statements revealing values (i.e., personal truths of individuals) and beliefs (i.e., cognitive statements to which the holder attributes truth or applicability) (Hannula, 2012) of the participating teachers regarding teaching geometry. The nature of these was (independent of the level) classified into three categories: positive (e.g., positive values such as 'important', 'indispensable' or positive beliefs such as 'Geometry is an essential part of mathematics teaching.'), negative (e.g., negative values such as 'secondary' or negative beliefs such as 'difficult', 'stressful') and neutral (Laine et al., 2015). By using two different coding systems (i.e., one reflecting the didactical potential of teaching geometry and the other reflecting the mathematics-related affect), it was possible to assign each statement to one of the developed categories.

In contrast to the open-ended question, the standardised item included 22 statements on the advantages of teaching geometry that have been taken from the work of Backe-Neuwald (2000) but also supplemented with statements from the more recent literature outlined earlier in the present paper with an option of writing an additional statement. Through these statements, all eight categories of the didactical potential of geometry were covered (i.e., motivation, sustainable learning environments, problem solving, opening up reality, spatial visualisation ability, acquisition of arithmetic concepts, geometric concept formation, and basic knowledge). The participants were asked to mark aspects of school geometry that were most relevant to them, and to hierarchise five answers accordingly (e.g., 1-most relevant). The item was evaluated according to the frequencies of all answer options and assigned rankings. Regarding the latter, a simple score was generated, which valued the most important reason with five points, the second most important with four points, and so on. The score, as opposed to merely looking at the absolute frequencies, allowed for a better assessment of the importance of the reasons, even though both metrics showed an almost equal ordering of the reasons. Both items complemented each other; by evaluating the open-ended item, teachers were given the opportunity to express themselves freely without being steered in one direction by predetermined answer choices. Through the additional evaluation of the standardised item, the spectrum of answers was expanded. It was therefore of particular interest whether the answers to the open-ended question corresponded to those of the standardised item, and to what extent new aspects emerged through the standardised item.

To determine the practical value of teaching geometry from the teachers' perspective, two items were analysed. Two items from section (2) 'characteristics of teaching geometry' focused on a quantitative aspect of teaching geometry, namely the number of lessons dealing with geometric topics per grade level, and on the form in which geometry instruction was anchored in mathematics instruction (i.e., parts of lessons, individual lessons, integrated into the weekly or daily schedule, integrated into interdisciplinary projects or series of lessons). In addition, when evaluating this item, the ranking undertaken by the teachers (e.g., 1-most often) was considered, which allowed individual items a higher weighting than others. Thus, it was assumed that teachers who checked the item 'series of lessons' and 'integration into the weekly or daily schedule' gave more space to geometry instruction than teachers who checked 'parts of lessons' or 'individual lessons'. The item 'integrated into interdisciplinary projects' indicated that geometry lessons were not considered as a part of regular mathematics lessons but were only treated in so-called project weeks. All standardised items were analysed using descriptive statistics.

Results

In this section, the results pertaining to the two research questions are presented. Concretely, I present the results regarding the didactical potential of teaching geometry recognised by the in-service teachers and the practical value they assign to teaching geometry.

Didactical Potential of Elementary School Geometry: Goals and Aspects

Here I present the results regarding teachers' associations with teaching geometry, and the benefits of teaching geometry that cover (2) 'aspects pertaining to characteristics of teaching geometry', and (4) 'goals and aspects of teaching geometry', respectively from the questionnaire. The results reflect a clear tendency regarding the teachers' responses to the open-ended item (see Table 1). Statements[2] regarding several categories are presented below along the

2 A subjective selection of content statements by the author is unavoidable; this has been handled in a comparable manner in the previous research (Backe-Neuwald, 2000), and does not necessarily diminish the validity of the conclusions.

variables of teaching experience, and professional background. The motivation category made up the largest part of the content-related statements with a total of 34 teachers (29.1%) stating that geometry lessons were highly motivating for the students. Their answers included statements such as:

- *'a way to get children interested in mathematics who otherwise do not like the subject.' (up to 2 yrs., specialist)*
- *'nice, because the students are motivated to do it.' (up to 2 yrs., non-specialist)*
- *'motivates the students, also those who don't like arithmetic so much.' (up to 20 yrs., specialist)*

In the second place of content-related statements, 12 teachers (10.3%) indicated sustainable learning environments, with a particular emphasis being assigned to the action-oriented aspect. Their answers included statements such as:

- *'very interesting, because it is so versatile and exciting, and a lot can be practically discovered, tinkered with and developed with the children.' (up to 2 yrs., specialist)*
- *'above all action-oriented work.' (over 20 yrs., specialist)*
- *'laying, folding, drawing, constructing, comparing, etc. practical, visualised, active learning.' (over 20 yrs., specialist)*

Even though in the third place of content-related statements, only 8.5% of the teachers' answers (n = 10) associated geometry teaching with the development of spatial visualisation ability. Their responses included statements such as:

- *'a lot of mental geometry, i.e., work on spatial visualisation ability ...' (up to 10 yrs., specialist)*
- *'a branch of mathematics, important for the development of spatial visualisation …' (over 20 yrs., specialist)*

All other content categories were coded to a limited extent, namely opening up reality, problem solving, acquisition of arithmetic concepts, geometric concept formation, and basic knowledge. Seven teachers did not fill out the item (5.8%).

Lastly, worth reporting are affect-related statements, which made up almost one-third of all statements. A total of 23.3% of the teachers (n = 28) had positive affect-related values and beliefs about teaching geometry. Occasionally, positive statements were associated with too little weight being given to geometry. The positive responses included statements such as:

- *'one of the most exciting and interesting subject areas, but nevertheless often underrepresented.' (up to 10 yrs., specialist, belief)*
- *'extremely important and a significant part of mathematics.' (over 20 yrs., specialist, value)*

Furthermore, eight affect-related statements (6.7%) reflected strong negative affect-related values and beliefs about teaching geometry. Negative responses included statements such as:

- *'secondary' (up to 2 yrs., non-specialist, value)*
- *'difficult because motor skills (holding pencil, drawing lines) are underdeveloped.' (up to 2 yrs., non-specialist, belief)*
- *'always a challenge.' (up to 20 yrs., specialist, belief)*
- *'stressful because I would have to have 20 hands to help everyone.' (over 20 yrs., specialist, belief)*

Table 1

Distribution of In-service Teachers' Free Associations Regarding Geometry Teaching

Type of category	Category	Absolute and relative frequencies
Content-related statement	Motivation	34 (28.3%)
	Sustainable learning environments	12 (10%)
	Problem solving	4 (3.3%)
	Opening up reality	7 (5.8%)
	Spatial visualisation ability	10 (8.3%)
	Acquisition of arithmetic concepts	4 (3.3%)
	Geometric concept formation	2 (1.7%)
	Basic knowledge	2 (1.7%)
Affect-related statement	Positive affect-related values and beliefs	28 (23.3%)
	Neutral affect-related values and beliefs	2 (1.7%)
	Negative affect-related values and beliefs	8 (6.7%)
No statement		7 (5.8%)

The results from the standardised item pertaining to the benefits of teaching geometry in primary school are shown in Table 2. Of 120 surveys, 119 teachers filled out this item. Additionally, not all participants ranked their answers (n = 24), so these data were not considered for the Top 5 Score. The teachers' answers predominantly confirmed the findings of the aforementioned open-ended item. Motivational aspects were included in several statements, such as 'is fun for students' (item 4.3.1), 'can promote a positive attitude towards the subject of mathematics' (item 4.3.17), and similarly was the most frequently coded content-related aspect. Likewise, the benefit of teaching geometry to support the acquisition of arithmetic concepts (item 4.3.19) was recognised by 11 teachers only. Here, in contrast, the relevance of spatial visualisation (item 4.3.13) predominated with 79.2% of teachers ($n = 95$) attributing this geometric goal the greatest relevance in primary school geometry. A total of 32 teachers attributed this statement one of the five possible rankings, with 18 of them giving it rank 1. Also, opening up reality (item 4.3.10), and problem solving (item 4.3.18) were attributed greater importance than was reflected on the open-ended item, with 39 and 41 instances, respectively.

Table 2

Benefits of Teaching Geometry in School Mathematics

Item	Statement Geometry in primary school …	Absolute frequencies (max.119)	Top 5 Score
4.3.13	promotes and supports spatial visualisation ability.	95	258
4.3.14	gives children who are otherwise weak in mathematics a sense of achievement.	73	126
4.3.6	offers children many opportunities to make independent discoveries.	69	127
4.3.1	is fun for students.	68	122
4.3.21	trains motor skills (e.g., sheathing, folding, stretching, drawing).	67	94
4.3.16	is an indispensable part of mathematics education.	59	84
4.3.2	promotes children's creativity.	53	59
4.3.3	promotes elementary mental abilities like ordering and classifying.	51	71
4.3.12	sharpens perception.	48	77
4.3.22	enables them to grasp content using concrete material.	47	50
4.3.18	makes an important contribution to the development of reality.	41	58
4.3.17	can promote a positive attitude towards the subject of mathematics.	39	50
4.3.10	enables a contribution to the formation of general mathematical competencies (e.g., reasoning, problem solving).	39	49
4.3.7	offers opportunities for open forms of teaching.	38	24

Item	Statement Geometry in primary school ...	Absolute frequencies (max.119)	Top 5 Score
4.3.9	promotes and challenges language competence.	35	31
4.3.20	lays the foundation for later systematic geometry teaching.	33	27
4.3.8	offers opportunities for interdisciplinary work.	33	26
4.3.4	enables individual differentiation.	25	25
4.3.15	promotes describing and uncovering structures.	24	15
4.3.11	promotes aesthetic sensibility.	22	10
4.3.5	promotes social skills.	20	10
4.3.19	supports the acquisition of arithmetic concepts.	11	7
4.3.23	Other:___________	0	0

Practical Value of Elementary School Geometry

Here I present the results pertaining to the practical value of elementary school geometry, specifically the form of instruction and teaching hours in geometry. Regarding the implementation form in which geometry lessons are taught, the results showed that they were mainly taught in the form of a series of lessons (65%), and individual lessons (64.2%) (see Figure 1). About one-third of participants indicated that parts of lessons, projects or daily/weekly schedules were filled with geometric content. In total, six participants did not fill out this item. Furthermore, only half of the teachers ranked the marked answers by frequency (n = 54) so that only a tendency could be depicted. Accordingly, 42.6% of teachers (n = 23) most often conducted a series of lessons, followed by 33.3% of teachers (n = 18) who conducted individual lessons. Here, the Top 5 Score was higher for individual lessons with a score of 186 than for a series of lessons with a score of 149 since more teachers gave ranks 1 and 2 to the former.

Figure 1
Different Implementation Forms of Geometry Lessons

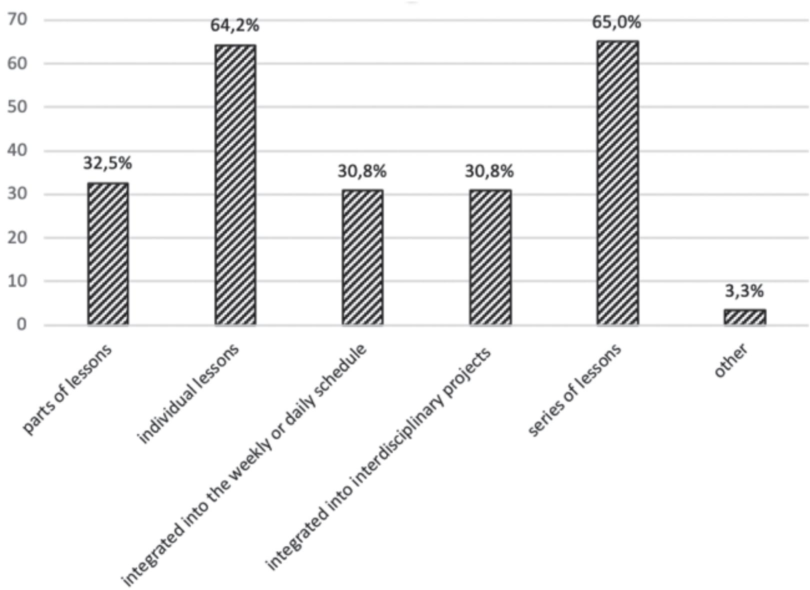

Table 3 illustrates results regarding the teaching hours per grade level within geometry instruction. It must be taken into account that there were missing data since the participating teachers did not necessarily teach mathematics in all grades. Furthermore, each federal state prescribes a contingent of hours in each subject within own elementary school policy regulations. Generally, these vary between 160 and 200 hours for mathematics. There are no concrete specifications as to how many hours and which areas should be covered. In the school's internal curriculum, this is mostly determined from school to school. Thus, the information on how many mathematics lessons and specifically geometry lessons were taught in total varies in every school. As an example, one participating school provided information on the number of geometry lessons per grade level: Grades 1–2 50 hours, Grade 3 50 hours, and Grades 4–6 70 hours. Lastly, some answers were given in percentages, and for that reason, could not be used and were unfortunately not included in the evaluation. Since these pieces of information were not available, only a tendency could be depicted; the number of geometry lessons in mathematics increases with the increasing grade level. However, similarities can be seen with respect to its extreme minimum value independent of the grade level.

Table 3
Distribution of Teaching Hours per Grade Level

	Grade					
	1	2	3	4	5	6
Average	15.77	18.83	20.74	25.76	28.29	34.23
Median	15	20	20	20	30	35
Standard deviation	9.416	9.903	11.710	16.239	15.332	18.390
Minimum	1	1	0	1	1	1
Maximum	35	40	60	55	60	80

Discussion

This study investigated the importance of school geometry in primary education by providing an insight into the extent to which both the didactical potential and the practical value of geometry instruction in elementary schools nowadays are recognised and utilised by primary grade teachers. Even though the importance of geometry instruction within elementary school mathematics increased significantly, as reported in the Backe-Neuwald's (2000) study as well as in newer replication and extension studies (Sitter, 2019; Wiese, 2016), the question about the current importance of geometry teaching cannot be answered unambiguously.

The results revealed that teachers employ teaching practices conducive to geometry learning but also revealed their insecurity regarding other important aspects as reported in the literature (Bauersfeld, 1992; Franke & Reinhold, 2016; Radatz & Rickmeyer, 1991; van den Heuvel-Panhuizen, 2008; Winter, 1997; Wollring, 2007). The evaluation of the open-ended item 'Geometry teaching in elementary school is for me …' showed an explicit subject didactic conception, especially for the topics of spatial visualisation, development of reality as well as problem solving. A total of 8.5% of the respondents associated spatial visualisation ability as a central aspect of teaching geometry in primary school. In the evaluation of the standardised items, item 4.3.13 'promotes and supports spatial visualisation ability' was ranked most significant with 95 respondents marking this statement. This result can be seen to be consistent with the study of Backe-Neuwald (2000) which was the third most common item chosen by teachers. Franke and Reinhold (2016), and Sinclair and Bruce (2015) also assign to this content one of the most important didactical potentials of geometry teaching in primary education; spatial visualisation is often listed as one of the main goals of primary grade geometry (Franke & Reinhold, 2016). On the one hand, spatial visualisation is found in many geometric topics, and, on the other, the topic of 'Space and Shape' is extensively anchored in the framework curriculum (RLP, 2015).

Other findings from this research suggest that geometry instruction is poorly linked to the rest of mathematics instruction. Although Graumann (2009), and Kuzle and Bruder (2016) make it clear that geometry instruction is particularly suited for training problem-solving abilities, few teachers reported this aspect. In the open-ended item, only 3.4% of the teachers commented on problem solving; in the standardised item, statement 4.3.10 on general mathematical competencies such as problem solving and reasoning, is ranked 13 out of 23 (n = 39). Similarly, this aspect had the least relevance in the study of Backe-Neuwald (2000). This is rather problematic, since problem-solving abilities are highly relevant for the whole mathematics education and can be learned and extended especially through geometry problems (Kuzle & Bruder, 2016). However, on a positive note, discovery learning, which is related to problem-solving skills in the broadest sense, was rated as a highly relevant aspect of geometry instruction. The fact that geometry 'offers children many opportunities to make independent discoveries' (item 4.3.6) was ranked 3 out of 23 (n = 69). This is aligned with Backe-Neuwald's (2000) study where this aspect was recognised as the most important aspect of teaching geometry by about 60% of respondents.

In Backe-Neuwald's (2000) study, discovery learning and opening up reality were ranked first and second as the most important aspects of geometry instruction, respectively. These can be seen as part of a successful geometry learning environment. The teachers surveyed referred to geometry learning environments in 10.3% of their responses (n = 12). According to Franke and Reinhold (2016), and Wollring (2007), geometry instruction is particularly suitable for creating sustainable learning environments. Concretely, teachers recognised differentiating instruction, and action orientation as important aspects of geometry instruction. These answers indicated that they dealt with individual aspects of successful learning environments. However, action orientation and differentiation were mentioned rarely on the open-ended item; 'enables individual differentiation' (item 4.3.4) and 'enables them to grasp content using concrete material' (item 4.3.22) appeared in 47 and 25 instances, respectively. Since the evaluation of the standardised item 4.3.6 showed that the statement 'offers children many opportunities to make independent discoveries' was highly ranked by the teachers, it can therefore be assumed that geometric learning environments are designed by teachers in the sense of discovery learning. However, it remains questionable to what extent these make a connection to the lived experiences of the students since (other than in the study of Backe-Neuwald (2000)) only about one-third of participants (*n* = 41) attributed importance to 'makes an important contribution to the development of reality' (item 4.3.18). Since it is through the connection to reality that children understand the relevance and the meaning of the content being learnt (Winter, 1976) it is worrying that this aspect has declined in relevance in the previous two decades.

The study results indicated that the teachers had a concrete idea of what geometry teaching was about and which aspects were considered to be particularly important (see Table 1). Nevertheless, the subject-specific responses only partially reflected the geometry discourse in the community since the answers on the open-ended item reflected the motivational aspect more (29.1%) rather than geometrical topics. Such association with geometry teaching can be further differentiated. One main argument for motivation was that children who are weak in arithmetic lessons can achieve a sense of achievement in geometry lessons and are therefore motivated. Also, in the evaluation of the standardised item, the statement 'enables children who are otherwise weak in mathematics to experience success' (item 4.3.14) was the second most frequently marked (*n* = 73; 61.3%). Thus, the teachers agreed with the statement of Krauthausen (2018, p. 105) that children can achieve a positive self-concept regarding mathematics through geometry instruction, creating an opportunity to compensate for weaknesses in arithmetic instruction. Even though this aspect was ranked fourth in the study of Backe-Neuwald (2000), it was more dominant since only about 30% of teachers (*n* = 32) agreed with this statement. Furthermore, teachers indicated that both they and the children enjoy geometry lessons. For instance, Graumann (1994) assumed that geometry instruction offered great potential for learning in an action-oriented, playful, and fun way. This view was also reflected in the teachers' statements as well as in the standardised item, with the statement, 'is fun for students,' (item 4.3.1) being marked fourth most often.

Furthermore, the survey supported the results of Backe-Neuwald (2000) that geometry is still barely connected to other areas of mathematics, such as arithmetic and algebra, which was revealed by both the open-ended item and the standardised item. The statement 'supports the acquisition of arithmetic concepts' (item 4.3.19) was hardly seen as relevant by the teachers (*n* = 11) or mentioned by the teachers on the open-ended item (*n* = 4). Furthermore, the statement 'promotes describing and uncovering structures' (item 4.3.15) was also recognised by 24 teachers only. These responses indicate that teachers treat geometry as an autonomous part of mathematics and make few references to arithmetic or algebra. These results clearly show that the two areas are still strongly separated. Additionally, the statement 'lays the foundation for later systematic geometry teaching' was selected least frequently by teachers (n = 33). This result indicates that geometry is still not being taught in the sense of a spiral curriculum, as Wittmann had called for it in 1999 but in a canonical way and not integrated into the network of different contextualised areas of competence. Wittmann (1999) spoke of the fact that 'geometry has no tradition in elementary school' (p. 208). Geometry can and must be the basis of arithmetic, both in terms of content and in the formation of mathematical teaching-learning processes (Bauersfeld, 1992; van den Heuvel-Panhuizen, 2008; Wittmann, 1999). Geometry is also useful in the subject area 'Patterns and structures' to understand and deepen arithmetic concepts, and to initiate algebraic structures (van den Heuvel-Panhuizen, 2008). If the often-lamented weak institutional, and also practical anchoring of geometry lessons in the curriculum is to be overcome, better integration with other mathematical subareas and competencies must be ensured. This is unfortunately not reflected in the current framework curriculum (RLP, 2015). A similar picture was reported by Glasnović Gracin and Kuzle (2018) regarding the designated curriculum in Croatia and students' performance. In order to link different mathematical areas, references between arithmetic, algebra and geometry should also be made within the framework curriculum. Also, in teacher training, greater emphasis on the subject and didactic interdependencies between geometry, arithmetic, and algebra instruction should be given (Franke & Reinhold, 2016).

The question of the practical value that in-service teachers assign to geometry instruction in elementary schools provided a much clearer picture. Geometry teaching offers a great potential to train and extend a wide variety of mathematical skills and abilities. The fact that 65% of the respondents conduct entire lesson series on geometric content and less than one-third teach geometry lessons in projects (30.8%) or as part of a lesson (32.5%) shows that geometry lessons have become highly valued. Thus, it seems that the teachers give geometric topics a high priority both in terms of content and time and that they have taken up a fixed place in mathematics lessons and are not outsourced in the form of a project but are an integral part of mathematics instruction. This is somewhat contradictory to results pertaining to the amount of time spent per grade level teaching geometry (see Table 3), but this item offers only a reliable result to a limited extent. However, there is a tendency that the number of teaching hours increases continuously from Grade 1 to Grade 6. Backe-Neuwald (2000) came to a similar conclusion, suggesting that more instructional time was needed for arithmetic basics in the lower grades and that geometry instruction was neglected for this reason. This result could be explained by the paradigm shift that started at the end of the 1990s. National and international movements within mathematics have led to a new emphasis on the subject, which has had a great influence on school life in general, but also on the teaching of geometry in elementary schools (Mammana & Villani, 1998) as well as on new educational standards (KMK, 2005). It remains problematic that Germany's educational system is famously decentralised, which means that Germany effectively has 16 different school systems, one for each federal state. Thus, the number of prescribed hours for each subject, and hence also mathematics differs. Also, each federal state's curriculum does not prescribe the number of hours for each mathematical topic which makes it difficult to understand the trend describing the teachers' choices.

Conclusions

The geometry didactic community attributes a high didactic potential to geometry instruction and a didactic and content-related dovetailing with the rest of mathematics instruction is demanded (Bauerfeld, 1993; Franke & Reinhold, 2016; Radatz & Rickmeyer, 1991; Winter, 1971; Wittmann, 1999). In recent decades, many goals and aspects of good geometry teaching have been elaborated, which were only partially implemented by the teachers surveyed. The study showed that the participating teachers had generally a positive attitude toward the teaching of geometry and a professional knowledge of basic topics of geometry didactics. However, it also became clear that the potential of geometry teaching was not being fully exploited by the teachers at the present time. Many of the geometry-relevant topics, goals and aspects were mentioned to a limited extent or were attributed minimal relevance. The positive attitudes and the willingness to deal with the topic field are optimal prerequisites for granting geometry didactics more attention. Many teachers stated that both they and their students are particularly motivated to deal with geometric content in the classroom. Essentially, some of the goals and aspects of good geometry teaching have already found their way into mathematics lessons. However, the results pertaining to the number of hours spent on geometry topics showed that the subject is still not treated equally compared to other areas of school mathematics.

This study was a mixed-methods study using convenience sampling. Thus, the participating teachers only represented the country to a limited extent. One should not forget that Germany's educational system is decentralised. Hence, the results of the study are limited to the curricula of the federal states of Berlin and Brandenburg. As such, the results may be limited to specific cultural and contextual characteristics. Also, due to voluntary participation, one may assume that the teachers were more motivated and that the results reflect the practices of motivated teachers. However, this is questionable since the data did not reveal a unanimous picture nor conclusive results in all cases. For the generalisability of the results in a wider setting, it is essential to recruit a larger sample from a variety of settings (e.g., federal states or countries) using alternative sampling strategies (e.g., maximum variation sampling, probability sampling), so that a researcher could create a less-biased and more thorough description of the current state of geometry teaching on both national and international levels, which can then be generalisable to a population.

The results have also provided evidence of possible theoretical as well as methodological biases. With respect to the former, an extensive literature review concerning modern geometry didactics was undertaken. However, surely not all aspects of its didactical potential have been covered but rather the main ones. With respect to the latter, some items were not entirely or fully answered by all participants (e.g., item 2.2), and some answers were not free of contradictions; for example, a majority of the respondents considered geometry teaching to be neglected and its topics to be unimportant in large parts, but nevertheless taught geometry with a positive attitude. Eliminating these problems with such an extensive and aspect-rich questionnaire is not likely to be trivial and would include the use of complementary instruments that are not based on self-assessment, such as student achievement tests, classroom documentation, or protocols as suggested by Wiese (2016). In that manner, one would obtain a more objective 'truth' by taking two perspectives in focus: the perspective of a teacher himself, and the perspective of the researcher. Furthermore, since this study showed that teachers were fundamentally positive about geometry instruction, and partially recognised its didactic potential, it would be interesting to further explore how teachers could be supported in conducting a high-quality geometry instruction that has an adequate place in mathematics education. Also, with the results of the questionnaire, it was difficult to make a statement about the number of hours geometry was taught in the individual grades. In this context, it would be interesting to investigate which concrete goals and contents are implemented and to what extent geometry is actually taught. By using the above-mentioned methods, it would be possible to make precise statements about the scope, goals, and content of geometry instruction, and to identify and analyse gaps.

By relating the study results to educational practice, some implications can be drawn as well as possible research of practical nature. Historically, educational policy, educational standards, and framework curricula have a strong influence on the status, goals, and content of school geometry (Franke & Reinhold, 2016). The strong influence of educational policy regulations on everyday teaching practice suggests that particularly relevant goals and content need to be more strongly anchored in framework curricula and standards. In the case of Germany with its decentralised system, it is questionable to what extent it is possible to set this as a goal as well as to prescribe hours for geometry topics. The existing curriculum models (e.g., Remillard & Heck, 2014) present the starting point for providing students' well-established opportunities to learn geometry. The designated curriculum (i.e., instructional plans specified by an authorised, governing body) here plays a vital role since it influences all components of the operational curriculum (i.e., teacher-intended curriculum, enacted curriculum, attained curriculum). Likewise, this may also affect countries with highly centralised educational systems when policymakers attribute geometry little attention within the designated curriculum. Also, at this point, it would be interesting to further research the extent to which teachers design their geometry instruction according to current framework curricula. It also remains to be critically questioned how well the paradigm shift in teacher education can be distinguished from the paradigm shift in schools over the previous two decades. To what extent has the paradigm shift been introduced in higher education institutions? Here, the questionnaire would need to be expanded by items focusing on the respective teacher training program which would involve a very extensive and time-consuming study.

The study showed that the teachers were basically positive about the geometry instruction and also partially recognised its didactic potential. That said, it would be interesting to further investigate how teachers can be supported in order to carry out optimal geometry teaching, which takes an adequate place in mathematics education. Adequate and solid teacher training focusing on both content and pedagogical content knowledge is essential as well as supplementary training in order to dissolve uncertainties when teaching geometry (Jones & Mooney, 2003). Thus, solid teacher training needs to better prepare future mathematics teachers to play the roles and to reflect the teaching practices conducive to geometry learning that have been emphasised in the literature. Thus, we need to understand how teachers may be better prepared to play the roles that have been emphasised in the literature as well as ongoing developments (Sinclair & Bruce, 2015). More generally, regular empirical studies on the status of elementary school geometry on different levels (e.g., school, university) should be conducted to obtain up-to-date developments. Only in this way can general statements be substantiated, and reality be depicted.

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