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Promoting Problem-Solving Skills by Engaging Students in Detecting, Explaining and Fixing Errors in Applications of the First Derivative in Individual and Collaborative Settings

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Learning from erroneous examples that involve step-by-step problem solutions containing errors that can be detected, explained and fixed by students could be beneficial for the students' problem-solving skills. Previous studies have investigated the effectiveness of erroneous examples in mathematics learning, but less attention has been focused on the effectiveness of the use of erroneous examples in individual and collaborative settings when erroneous examples are combined with self-explanation prompts and practice problems addressing students' problem-solving skills. The present quasi-experimental study with a post-test only non-equivalent group design was therefore intended to examine the extent to which presenting erroneous examples in individual and collaborative settings could promote students' problem-solving skills. The results suggest that the use of erroneous examples in both settings is effective in promoting students' problem-solving skills, with neither setting being better than the other. In light of these results, teachers can vary the use of these learning settings in facilitating their students' learning through erroneous examples.

Keywords: erroneous examples, problem-solving skills, individual setting, collaborative setting

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Spodbujanje spretnosti reševanja problemov z vključevanjem učencev v odkrivanje, pojasnjevanje in v odpravljanje napak pri uporabi prve izpeljanke v individualnem in sodelovalnem okolju

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Učenje iz napačno rešenih primerov, ki vključujejo reševanje problemov po korakih, pri čemer so prisotne napake, ki jih lahko učenci odkrijejo, pojasnijo in popravijo, bi lahko koristilo učencem pri poglabljanju njihovih spretnosti reševanja problemov. V prejšnjih študijah je bila raziskana učinkovitost napačno rešenih primerov pri učenju matematike, vendar je bilo manj pozornosti namenjene učinkovitosti uporabe napačno rešenih primerov v individualnem in sodelovalnem okolju, ko so napačno rešeni primeri kombinirani z napotki za samopojasnjevanje in s praktičnimi nalogami, ki naslavljajo spretnosti reševanja problemov pri učencih. Namen te kvazieksperimentalne študije z neekvivalentno zasnovo skupin, ki je bila izvedena le po testiranju, je bil zato preučiti, v kolikšni meri lahko predstavitev napačno rešenih primerov v individualnem in sodelovalnem okolju spodbuja spretnosti reševanja problemov pri učencih. Izsledki kažejo, da je uporaba napačno rešenih primerov v obeh okoljih učinkovita pri spodbujanju spretnosti reševanja problemov pri učencih, pri čemer nobeno okolje ni boljše od drugega. Glede na te izsledke lahko učitelji pri spodbujanju učenja učencev s pomočjo napačno rešenih primerov različno uporabljajo ti obliki učnega okolja.

Ključne besede: napačno rešeni primeri, spretnosti reševanja problemov, individualno okolje, sodelovalno okolje

Introduction

Problem-solving skills have been extensively recognised as essential skills that students need to develop through mathematics learning. Mink (2010) has argued that critical thinking and problem-solving skills are two of a range of significant skills that students need to develop to prepare themselves for the specific challenges and opportunities of twenty-first century society. In solving mathematics problems, students may experience difficulties due to several factors, including the complexity of the problem solving process (Mink, 2010), a lack of understanding of mathematical objects, facts, principles, concepts or procedures (Rumasoreng & Sugiman, 2014; Tias & Wutsqa, 2015), and a lack of mastery of various mathematical skills (Tambychik & Meerah, 2010). The difficulties that students experience in the problem-solving process can lead them to make errors (Rafi & Retnawati, 2018) that prevent them from solving the problem. Olivier (Ganesan & Dindyal, 2014) argued that students make errors when they are not careful in formulating a problem-solving strategy. These errors can then reoccur repeatedly when dealing with similar problems due to the students' misconceptions, which arise from a failure to accommodate and assimilate knowledge during the learning process (Sarwadi & Shahrill, 2014).

Example-based learning is believed to be one of the learning models that teachers can apply to help their students reduce and overcome misconceptions and encourage their problem-solving skills. Example-based learning is a learning model that employs an example as the basis of the learning activity in order to show students how to apply particular theories, concepts or formula in a certain context (Sern et al., 2015). Huang (2017) proposes four types of examples that can be presented to students when teachers apply example-based learning in their classroom: standard worked examples, erroneous worked examples, expert (masterly) modelling examples, and peer (coping) modelling examples. Erroneous examples (also known as erroneous worked examples or incorrect worked examples) is a problem accompanied by its solution presented in a step-by-step form and designed so that the solution contains errors with a specific intention (Isotani et al., 2011; Tsovaltzi, Mclaren, et al., 2010; Tsovaltzi, Melis, et al., 2010; Zhao & Acosta-Tello, 2016). Through presenting erroneous examples, students are encouraged to detect, explain and fix the existing errors as a means for knowledge acquisition and construction.

In determining what type of examples to present to students in order to maximise the learning benefits, teachers need to consider the students' characteristics. In learning mathematics, students must activate their prior knowledge, which will assist them in understanding new information and determine the direction of their thinking about the topics or competencies they are learning (Bruning et al., 2010). Students' prior knowledge is therefore a key characteristic. When engaging students in example-based mathematics learning, it has been found that correct worked examples are more favourable for those with poor prior knowledge, but less valuable for those with high prior knowledge (Große & Renkl, 2007). However, a study of Fitzsimmons et al. (2021) found that students with high prior knowledge also benefit from correct worked examples.

Some studies indicate that not all students benefit from learning mathematics through working on erroneous examples, suggesting that only students with high prior knowledge benefit from such examples, in which they perform better than students with low prior knowledge (Durkin & Rittle-Johnson, 2012; Große & Renkl, 2007; Hartmann et al., 2021; Heemsoth & Heinze, 2014; Zhao & Acosta-Tello, 2016). However, other studies have demonstrated that students with low prior knowledge can also benefit from the presentation of erroneous examples (Adams et al., 2012; Barbieri & Booth, 2016). Students with high and low prior knowledge can thus benefit to some degree from their engagement in learning activities that use correct worked examples and erroneous examples.

Teachers are generally still hesitant to present erroneous examples to their students because they are worried that it may lead to misconceptions about the material they are learning (Heemsoth & Heinze, 2014; McGinn et al., 2015; Rushton, 2018; Zhao & Acosta-Tello, 2016). Bridger and Mecklinger (Yang et al., 2016) find that erroneous examples tend to be avoided in conventional learning because teachers assume that such examples may lead students to reproduce the same errors when they are solving a similar problem, and that they do not benefit students' long-term memory. However, there is a lack of evidence regarding these concerns (Heemsoth & Heinze, 2014). Presenting erroneous examples makes it possible for teachers to encourage their students to think about each step in the problem-solving process, to determine why a particular step is considered to be an error, and to avoid misconceptions or errors in understanding or solving the problem in the future (Lange et al., 2014). As VanLehn (Heemsoth & Heinze, 2014) points out, constructivists believe that erroneous examples represent a huge opportunity for students to learn and to develop a better understanding. A number of previous studies have also demonstrated that students benefit from presenting erroneous examples in terms of promoting conceptual understanding (Booth et al., 2013; Tsovaltzi, Mclaren, et al., 2010), procedural understanding (Große & Renkl, 2007; Yang et al., 2016), metacognitive skills (Tsovaltzi, Mclaren, et al., 2010; Tsovaltzi, Melis, et al., 2010), problem-solving skills (Chen et al., 2016), a positive attitude towards errors in problem solving (Yang et al., 2016) and mathematical disposition (Rafi & Setyaningrum, 2019).

Two learning settings can be used by teachers to support the implementation of a particular learning model: an individual setting or a small group setting. The small group setting can be divided into two types: cooperative and collaborative. In an individual setting, every student has the right to do the task assigned by the teacher, or to learn a specific competence by applying their own knowledge, understanding or ability without having to consider the opinions of other students (Sultan et al., 2011). In a cooperative type of small group setting, students are facilitated to work together in a small group to complete a product or final goal. In a collaborative type group setting, on the other hand, students are facilitated to work in a small group in which each group member has responsibility for their own work, while embracing mutual learning and respect for the abilities and contributions of other group members (Panitz, 1999). By implementing erroneous examples in a small group setting, students are facilitated to discuss the examples in such a way that the group members help each other in detecting, explaining and correcting the errors, thus gaining an understanding of concepts and procedures.

Presenting erroneous examples to students in mathematics learning

Erroneous examples, also known as incorrect worked examples, are problems presented with a step-by-step procedure to find a solution, but where one or more steps contain errors with a specific intention (Isotani et al., 2011; Tsovaltzi, Mclaren, et al., 2010; Tsovaltzi, Melis, et al., 2010; Zhao & Acosta-Tello, 2016). Based on several studies related to the use of erroneous examples in mathematics learning (Durkin & Rittle-Johnson, 2012; Heemsoth & Heinze, 2014), it is essential to provide introductory instructions before presenting an erroneous example to students. These instructions are intended to build the students' knowledge of important terms in the learning material, or to provide a brief explanation of the material that they will study further in the erroneous examples. Previous studies have applied erroneous examples in mathematics learning to various topics encountered from the primary school to the university level, such as decimals (Durkin, 2012; Durkin & Rittle-Johnson, 2012; Isotani et al., 2011), fractions or proportions (Heemsoth & Heinze, 2014, 2015), algebra (Barbieri & Booth, 2016; McGinn et al., 2015; Zhao & Acosta-Tello, 2016), and subtraction on integers (Yang et al., 2016). These topics have been chosen because students often experience misconceptions or make errors in understanding the concepts or procedures they contain. This is in line with the finding of Booth et al. (2015) that erroneous examples should ideally present mistakes that are very commonly made by students when learning to solve a particular type of problem. In their research, however, Große and Renkl (2007) chose the topic of probability to present erroneous examples, even though there were no misconceptions or errors frequently made by students when studying probability.

In practice, the use of erroneous examples in mathematics learning can be applied in both individual (Barbieri & Booth, 2016; Heemsoth & Heinze, 2014) and collaborative (Yang et al., 2016) settings. In general, most studies choose a combination of erroneous and correct examples in mathematics learning (Durkin, 2012; Große & Renkl, 2007; Yang et al., 2016). Using a combination of erroneous and correct worked examples in mathematics learning is more effective than only using a combination of correct worked examples (Durkin, 2012; Durkin & Rittle-Johnson, 2012). The use of a combination of correct worked examples in Durkin's (2012) and Durkin and Rittle-Johnson's (2012) studies was in the context of decimal positioning on a number line with the interval from 0 to 1, with the reasoning behind the decimal positioning given by two different people. The combination of two correct examples in these studies led to the correct positioning of a decimal on the number line and the explanations for the decimal positioning given by the two people were also correct. The intention of presenting two different correct worked examples was for the students to be able to explain the difference in perspectives on the reasons for the decimal placement on the number line as well as the reasons why the two perspectives are both correct. Siegler (Booth et al., 2013) states that erroneous examples can help students to develop their procedural knowledge through activities to detect errors in such examples, and that students' conceptual knowledge and procedural knowledge can increase through the activity of recognising errors and comparing them with correct examples (Booth et al., 2013). According to Wang et al. (2015), learning with erroneous examples can cause intrinsic and extrinsic cognitive load, while students who learn with such examples have a higher germane cognitive load than students who learn with correct examples. Intrinsic cognitive load and extrinsic cognitive load arise because students are asked to detect, explain and correct existing errors when learning with erroneous examples. However, the activities of detecting, explaining and correcting errors can also support the development of students' germane cognitive load (Adams et al., 2012).

Engaging students to learn mathematics in individual and collaborative settings

Each student has different characteristics in terms of learning style and ability to learn certain learning content and skills. Teachers can implement an individual learning setting to facilitate the differences in student characteristics. In this setting, students strive independently to build their knowledge through the means provided by the teacher in the form of applying models, approaches, strategies or methods of learning. In an individual learning setting, students are facilitated to complete tasks or understand something based on their own knowledge, understanding or abilities, without having to consider the opinions of other students (Kertu et al., 2015; Sultan et al., 2011). Under certain conditions, learning in an individual setting is more beneficial for students than learning in a group setting. When students learn in an individual setting, they tend to complete their work on their own or ask the teacher for help when they encounter difficulties. However, when students in an individual setting experience problems in learning something or solving a problem, their interest and attention may decrease due to reduced opportunities for communication in discussions between students (Kertu et al., 2015).

Teachers' concerns about the unfavourable impact of continuously learning in individual settings can be reduced by making learning settings more varied, including the implementation of collaborative or group learning settings. MacGregor (Laal & Laal, 2012) argued that such learning settings allow students to solve a problem or task or create a product by collaborating with each other. In addition, in collaborative or group learning settings, students interact with each other to discuss a concept, debate differences of opinion regarding the concept and evaluate the results of their discussion (van Boxtel et al., 2000). Through such learning settings, students are also facilitated to (1) develop critical thinking, analytical thinking and communication skills; (2) work together effectively; and (3) appreciate the ideas and problem-solving strategies that other students propose (Sofroniou & Poutos, 2016). Furthermore, implementing a group work or collaborative setting facilitates students to be more confident, to overcome the fear of making mistakes, to reduce mathematics anxiety, to develop a positive attitude towards mathematics, and to be successful in academic life (Koçak et al., 2009).

Problem-solving skills in mathematics learning

Krulik and Rudnick (1988) define a problem as a condition for which a solution needs to be found, whereby the way to find the solution cannot be immediately known. Thinking skills including synthesising prior knowledge are required to arrive at the desired solution. Presenting erroneous examples to students is believed to be a potential strategy for teachers to encourage their students to grasp standard problem solving in mathematics learning. The rationale behind this belief is that through learning that includes erroneous examples, students engage in learning activities that involve detecting, explaining and fixing errors that are expected to help them in applying, adapting, monitoring and reflecting on a problem-solving process based on their knowledge and skills. By using the knowledge and skills they already have to solve a problem, students gain new experiences. Thus, presenting erroneous examples to students in mathematics learning is expected to contribute to the growth of students' problem-solving skills.

Students can use various models to solve a problem, especially a mathematics problem. The distinguishing feature of the models is the number of steps (three, four or five) needed to obtain a solution. A prominent four-step problem-solving model is that introduced by Polya (1985), which involves understanding the problem, identifying possible plans to solve the problem, executing the plan with the greatest problem-solving potential, and checking every step of the problem-solving process as well as the answer obtained. The last step is intended to ensure that the problem-solving procedure does not contain errors and the answer obtained is the solution to the problem. Another model that can be used in problem solving consists of a sequence of five steps: identifying the problem, representing the problem, choosing the appropriate strategy, implementing the chosen strategy, and evaluating the solution obtained (Bruning et al., 2010). Succeeding in identifying problems requires creativity, persistence and a desire to ponder the problem, whereby identifying the problem is considered the most difficult and challenging step in solving mathematics problems (Bruning et al., 2010). As emphasised by Fadillah (2009), students also have difficulty solving mathematics problems if they struggle with representing the problem. Thus, identifying and representing mathematics problems are considered to be significant problem-solving skills.

By adapting several problem-solving models, Tambychik and Meerah (2010) arrived at three steps to solve a problem: reading and understanding the problem; planning possible strategies that can be used to solve the problem and solving the problem using the chosen strategy; and confirming the problem-solving process and the answer. Based on the aforementioned three models, we believe that problem solving involves a complex set of skills comprising understanding the meaning of a given problem, developing a problem-solving plan, executing problem solving based on the developed plan, and confirming the first three steps to ensure that there are no errors and that the answer obtained is the solution to the problem.

Research problem and research questions

Problem solving has been recognised as a focus in school mathematics learning, both as a skill and as a method to facilitate students' learning of mathematics. Given that problem solving is an essential skill that can be facilitated through mathematics learning, extensive studies have explored various methods or strategies that have great potential in promoting this skill across educational levels and topics in school mathematics. Exploration of these various methods or strategies can be undertaken by considering factors that may contribute to students' problem-solving success, one of which is the ability to detect concepts relevant to the problem-solving strategy and connect these concepts (Antunović-Piton & Baranović, 2022). Among the methods or strategies that promote problem-solving skills, engaging students in reading mathematical texts related to inductive reasoning is an alternative strategy that can be applied in mathematics learning to support problem-solving skills based on the four-phase model proposed by Polya (Papadopoulos & Kyriakopoulou, 2022). Problem-solving skills can be developed through such a strategy because it allows students to interpret, evaluate and assimilate the mathematical texts they read (Papadopoulos & Kyriakopoulou, 2022). The present study is complementary to that of Papadopoulos and Kyriakopoulou (2022) in terms of contributing to demonstrating the great potential of engaging students in reading mathematical tasks that not only strengthen concept understanding, but also promote problem-solving skills. By considering the factors that influence problem-solving success in mathematics, as reported by Antunović-Piton and Baranović (2022), we extend the application of the method or strategy of reading mathematical tasks through the use of erroneous examples that engage students in understanding mathematical concepts not only by reading the problem-solving process, but also by identifying, explaining and correcting the errors contained in it.

Multiple studies have shown that the use of erroneous examples in mathematics learning has mostly been carried out in an individual setting (Barbieri & Booth, 2016; Heemsoth & Heinze, 2014; Zhao & Acosta-Tello, 2016). Presenting erroneous examples in a collaborative setting therefore needs more investigation. The use of erroneous examples in learning mathematics with individual and collaborative settings where the two groups of learning settings come from different classes also needs to be explored. Although a number of studies have been conducted, especially in the cognitive domain, to determine the effect of using erroneous examples in learning mathematics either presented alone (Heemsoth & Heinze, 2014; Tsovaltzi, Mclaren, et al., 2010; Tsovaltzi, Melis, et al., 2010) or combined with correct examples (Durkin & Rittle-Johnson, 2012; Große & Renkl, 2007; Yang et al., 2016; Zhao & Acosta-Tello, 2016), little attention has been paid to the effectiveness of erroneous examples in individual and group settings in terms of students' problem-solving skills, especially if the erroneous examples are designed according to the scheme suggested by McGinn et al. (2015), who introduced erroneous examples with combined similar problems that the students can solve after detecting, explaining and correcting the errors in the erroneous examples.

The present study focuses on examining the effectiveness of presenting erroneous examples to students in individual and collaborative settings that enable them to detect, explain and correct existing errors in their mathematical problem-solving skills. Specifically, the study attempted to answer the following research questions (RQs):

RQ1: Is the use of erroneous examples that require students to detect, explain and fix errors in problem solving in an individual setting effective in facilitating students' problem-solving skills?

RQ2: Is the use of erroneous examples that require students to detect, explain and fix errors in problem solving in a collaborative setting effective in facilitating students' problem-solving skills?

RQ3: Is there a difference in the effect on students' problem-solving skills due to the different learning settings applied when using erroneous examples in mathematics learning?

Based on a literature review, the proposed hypothesis is that involving students in learning activities in the form of detecting, explaining and correcting errors contained in erroneous examples and solving similar problems could effectively support student problem solving regardless of the learning setting that teachers implement. In addition, we also hypothesise that students who learn with erroneous examples in a group setting outperform those who learn in an individual setting.

Method

Participants

The sample comprised two of nine classes from grade 11 in the Mathematics and Science programme at a public senior high school in Yogyakarta City, Indonesia. Each of the two classes consisted of 26 students aged around 17 years. The students from the first class were facilitated to learn mathematics by presenting erroneous examples in an individual setting (hereinafter referred to as the individual group), while those from the second class were facilitated to learn mathematics by presenting erroneous examples in a group or collaborative setting (hereinafter referred to as collaborative group). After applying the two different treatments, the students were tested to determine the extent of their skills to solve problems related to the topics they had studied, i.e., the application of the first derivative.

Instrument

After studying applications of the first derivative by working on erroneous examples, the students took a problem-solving skills test on that topic. The test was administered to the students in the individual group three days after the last meeting for learning activities, and to students in the collaborative group seven days after the last meeting for learning activities. The test administration schedule was adjusted to the class schedule of each group. All of the students, in both the individual and the collaborative groups, were given 60 minutes to work on the test individually. The test consisted of four constructed-response items that are appropriate for use based on qualitative assessment by a mathematics education expert who holds a doctoral degree. This item format was chosen because it takes less time to create, it minimises students having to guess the answer, and it allows students to analyse and synthesise information (Reynolds et al., 2010; van Blerkom, 2009). The test was presented on four sheets, with each sheet consisting of one test item and space for the students to write their responses. The students were required to respond to each test item by writing down what is given in the question, what is asked, the step-by-step solution, and a conclusion on the solution they obtained. This was done in order to be in line with the operational definition of problem-solving skills established in this study, and to be consistent with the way student work was assessed in the test. The following are the four items contained in the problem-solving skills test.

Problem 1. It is known that the costs required to produce x units of a good are expressed by the function p that is defined as $p(x) = 10x^2/3 - 100x + 1000$ (in tens of thousands of rupiah), while the selling price of goods per unit is expressed by $j(x) = 50 - 5x/3$ (in tens of thousands of rupiah). Determine the maximum profit that can be obtained if all x units of the goods produced can be sold.
Problem 2. Determine the equations of the tangent and normal lines to the curve of the function f defined by $f(x) = x^3 + 2x^2 - 5x$ at $(-1,6)$.
Problem 3. Given a function f defined as $f(x) = x^3 + 3x^2 - 9x - 7$ determine the intervals of x on which the curve given by the function f is:
a. Always increasing
b. Always decreasing
Problem 4. Determine maximum turning point and minimum turning point of the curve of the function defined by $f(x) = x^3 + 3x^2/2 - 6x - 2$.

It was found that the use of this test item format brings two challenges: firstly, a great deal of time is required to assess the student work and, secondly, it is difficult to obtain highly reliable estimates of test scores due to the potential for subjectivity in assessing the students' work. A holistic scoring rubric was created to minimise subjectivity in scoring the students' work in the test and to obtain adequate estimates of test score reliability. In this way, the reliability estimates of the problem-solving skills test scores were adequate (Cronbach's $\alpha = 0.670$, McDonald's $\omega = 0.679$) (Reynolds et al., 2010; Rudner & Schafer, 2002; Wells & Wollack, 2003). The score that the students obtained in the test is the sum of the scores they obtained in the four existing test items. The maximum score on each test item was 16 points, so the maximum overall score was 64 points. The maximum score for each test item is the sum of the scores for the responses to the test item based on four scoring aspects: understanding the problem given, planning to solve the problem, executing the determined problem-solving plan, and confirming the solution to the problem. Each aspect has an integer score ranging from 0 to 4 points.

Research design

In order to answer the research questions, a quasi-experiment method with a post-test only non-equivalent design was employed. This method was chosen to fit the school setting or circumstances, which did not allow us to randomly select students and then randomly assign them to research groups. Furthermore, the design matched the main objective of the study, which was to compare the effectiveness of two different treatments on two different groups that were only assessed based on one measure after each group received the treatment. We declare that this study was conducted in accordance with applicable research ethics and with permission obtained from the school where the study was carried out, the Faculty of Mathematics and Natural Sciences, Universitas Negeri Yogyakarta, and the Department of Education, Youth and Sports of the Regional Government of the Special Region of Yogyakarta, Indonesia.

Procedure

The study took place in five meetings for each group: the first four meetings were used for learning activities, and the last meeting was used for administering the problem-solving skills test. The meetings lasted 90 minutes and each meeting for the learning activities focused on its own topic. The topic of the first meeting was the application of the first derivative to determine the (local) minimum and maximum of a curve of a function. At the second meeting, the learning activities focused on the application of the first derivative to determine the equation of the tangent and normal to a curve of a function. At the third meeting, the focus was on the application of the first derivative to find increasing and decreasing functions, while the fourth meeting focused on facilitating students to learn about identifying relative extrema and sketching the curve of a function using the first derivative. The learning activities for both groups at the four meetings were facilitated by the first author of the present paper and carried out according to lesson plans developed in accordance with the treatment designed for each group. The two treatment groups in the study were facilitated to learn the four aforementioned topics equally, using the same erroneous examples worksheets in terms of both design and content.

The erroneous example-based worksheet presented to the students was developed based on errors that students commonly make in solving problems related to applications of the first derivative, as revealed by a number of previous studies (Hadi, 2012; Putri, 2013; Sakti, 2017). Sarwadi and Shahrill (2014) state that errors made by students can reflect their understanding of mathematical concepts, problems or problem-solving procedures. Errors made by students when solving problems can be in the form of factual errors caused by a lack of understanding of factual information such as mathematical terms; procedural errors that represent errors in carrying out the steps of mathematical problem solving; conceptual errors caused by misconceptions or errors in understanding of mathematical facts and principles; and careless errors caused by fatigue or irrelevant distractions when solving problems (Brown et al., 2016). Below are examples of errors made by students when solving problems related to the application of the first derivative with the context of determining maximum profit, as identified in a study by Putri (2013), which was used as one of the references in developing the erroneous examples in the present study.

A textile company produces x pairs of jeans at a total cost of $20x - 75 + x^2$ thousand Indonesian Rupiah (IDR). If all the jeans are sold at a price of IDR 100000, for each pair of jeans, what is [the maximum] profit that the company [can] earn?

Student's Response

$20x - 75 + x^2 \rightarrow (x^2 + 20x - 75)$ thousand rupiah [*error in understanding the intent of the problem and in devising a plan to solve the problem; the objective function should be a function of profit, where profit represents the income from the sale of x pairs of jeans after subtracting the cost to produce x pairs of jeans: $f(x) = (100x - (20x - 75 + x^2))$ thousand rupiah $= (80x + 75 - x^2)$ thousand rupiah*]

$f' = 2x + 20$ thousand rupiah [*Given that the objective function should be $f(x) = (80x + 75 - x^2)$ thousand rupiah, the maximum profit can be earned when $f'(x) = 80 - 2x = 0$*]

Number of jeans $= 20x + 20 = 0 \leftrightarrow 2x = -20 \leftrightarrow x = -20/-2 = 10$ [*error in performing mathematical operations. The maximum profit can be earned when $f'(x) = 80 - 2x = 0$ or $x = 40$. This means that it needs to sell 40 pairs of jeans to earn maximum profit*]

Production cost $= x^2 + 20x - 75 = (10)^2 + 20(10) - 75$ thousand rupiah $= 100 + 200 - 75$ thousand rupiah $= 225$ thousand rupiah

Profit $= 10(100)$ thousand rupiah $- 225$ thousand rupiah $= 1000000 - 225$ thousand rupiah $= 675000$ thousand rupiah [*Overall, the students made an error in modelling the profit function or not understanding the meaning of profit when it is associated with production cost and the selling price. The maximum profit $= (80x + 75 - x^2)$ thousand rupiah $= (80(40) + 75 - (40^2))$ thousand rupiah $= 1675$ thousand rupiah*]

We followed the steps of developing an erroneous example-based worksheet suggested by McGinn et al. (2015). The first step in developing the worksheets containing erroneous examples and practice problems was to determine the target to be achieved by presenting the problem-solving error to the students and listing several common misconceptions or errors (e.g., factual, conceptual, procedural or careless errors) encountered in achieving this target. One misconception or error was then selected for each example and erroneous examples were created based on the selected misconception or error. The erroneous examples were then complemented with questions as self-explanation prompts based on existing misconceptions or errors in order to help the students identify, understand, analyse and fix their existing misconceptions or errors in the problem-solving steps. Through these activities, the students are expected to learn from these errors and not make similar errors when solving a problem. In the final step, problems were created that are similar to the problems proposed in the erroneous examples. This similar problem was intended as practice for the students, whereby the only difference between the problems in the erroneous examples and the practice problems is the number. Figures 1 and 2 show the erroneous examples and the corresponding practice problems that were presented to the students. The error types that were the focus of the erroneous examples were not presented; the error types shown in Figures 1 and 2 are only used to denote the error types that were included in the erroneous examples.

Figure 1

One of the erroneous examples presented to the students focused on a conceptual error

Find the value of the first derivative of $g(x) = 3x^2 - 2x - 7$ when $x = \frac{1}{3}$!

Solution

Known: $g(x) = 3x^2 - 2x - 7$

Asked: $g'\left(\frac{1}{3}\right) = \ldots?$

Answer:

$g(x) = 3.(2-1)x^{2-1} - 2 = 3x - 2$ [**conceptual error**]

$g'\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) - 2 = 1 - 2 = -1$

So, the value of the first derivative of $g(x) = 3x^2 - 2x - 7$ when $x = \frac{1}{3}$ is -1.

In the step-by-step problem solving above, which step or part is incorrect?

Is it true that the value of the first derivative of $g(x) = 3x^2 - 2x - 7$ when $x = \frac{1}{3}$ is -1? If it is not, then what is the maximum profit? Explain!

Practice Problem

Find the value of the first derivative of $g(x) = -4x^2 + x - 3$ when $x = -\frac{1}{4}$!

Figure 2

One of the erroneous examples presented to students focused on a careless error

It is known that the production cost of x radios per day is $\left(\frac{1}{4}x^2 + 35x + 25\right)$ tens of thousands rupiah, while the selling price of each radio is $\left(50 - \frac{1}{2}x\right)$ tens of thousands rupiah. Determine the maximum profit that can be obtained if all x radios produced can be sold!

Solution

Suppose that

$B(x)$: the production cost of x radios per day

$J(x)$: the selling product of each radio, and

$K(x)$: the profit that can be obtained if all x radios produced can be sold.

Known: $B(x) = \frac{1}{4}x^2 + 35x + 25$ and $J(x) = 50 - \frac{1}{2}x$

Asked: K_{maximum} (the maximum profit that can be obtained)

Answer:

profit = number of radios sold × selling price of each radio – the production cost of radios sold

$$
\begin{aligned}
K(x) &= xJ(x) - B(x)\\
&= x\left(50 - \frac{1}{2}x\right) - \left(\frac{1}{4}x^2 + 35x + 25\right)\\
&= 50x - \frac{1}{2}x^2 - \frac{1}{4}x^2 - 35x - 25\\
&= -\frac{3}{4}x^2 + 15x - 25
\end{aligned}
$$

$K'(x) = -\frac{3}{2}x + 15$

The profit will be maximum when $K'(x) = 0$

$K'(x) = 0 \Leftrightarrow -\frac{3}{2}x + 15 = 0 \Leftrightarrow -\frac{3}{2}x = -15 \Leftrightarrow x = 10$

(the profit will be maximum when the number of radios sold is 10)

$x = 10 \Leftrightarrow K(10) = -\frac{3}{4}(10)^2 + 15(10) - 25 = 75 + 150 - 25 = 200$ **[careless error]**

So, the maximum profit that can be earned is 200 tens of thousands rupiah.

In the step-by-step problem solving above, which step or part is incorrect?

Is it true that the maximum profit earned is 200 tens of thousands rupiah? If it is not, then what is the maximum profit? Explain!

Practice Problem

To produce x units of an item in one day is $(x^2 + 3x - 10)$ hundred thousand rupiah, while the selling price of the item per unit is $(19 - x)$ hundred thousand rupiah. Determine the maximum profit that can be obtained if all units of items produced can be sold!

In the present study, the number of items of pairs of erroneous examples and corresponding practice problems presented to the students were as follows: five items in the first meeting, five items in the second meeting, four items in the third meeting, and two items in the fourth meeting. The number of items was adjusted to the complexity of the concept, principle or procedure being studied at the learning meeting. In addition, the complexity of the erroneous examples in each learning meeting was gradually graded from low to high. In the first meeting, the topic of the erroneous examples began with errors made by the students when determining the value of the first derivative of a function (see Figure 1) and determining the (local) maximum and minimum values of a function, and extended to determining the optimum value of a function in the context of real-life problems using the first derivative (see Figure 2). In the second meeting, the topics of the erroneous examples started from determining the slope of the tangent line of a curve at a point, determining the slope of the normal line of a curve and determining the equation of the tangent and normal lines of a curve, and extended to determining the equation of the tangent line of a curve parallel and perpendicular to a given line. In the third meeting, the topics of the erroneous examples were determining the intervals where a function is increasing or decreasing and the point and type of stationary. In the last meeting, the erroneous examples focused on determining the inflection point and sketching the curve using the first derivative. It should be emphasised once again that each erroneous example focused on only one type of error: a factual error, a conceptual error, a procedural error or a careless error.

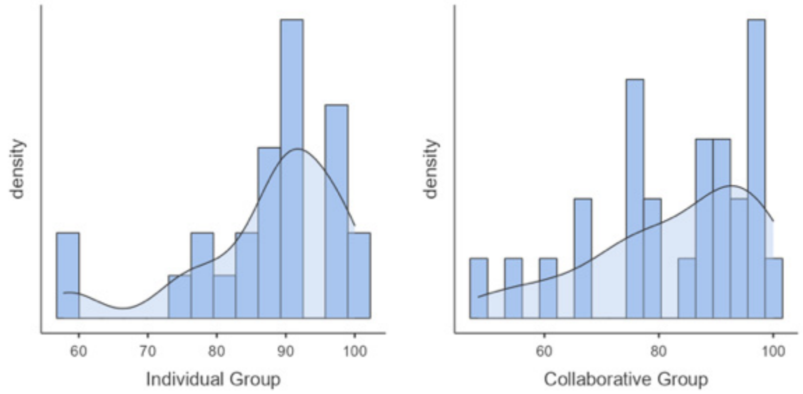
The learning process in the individual group began by providing brief information regarding the material that the students would learn. This was done for about 10–15 minutes in order to teach the basic terminology or concepts that the students needed to know before working on the worksheets that had been distributed to them. The students were then allowed to detect, explain and fix the errors in the erroneous examples and work on the practice problems on the worksheets. The learning process in the collaborative group was generally the same as in the individual group, except that the students carried out the activities to detect, explain and fix the errors in the erroneous examples in groups and through group discussions. Each group consisted of three or four students, with the group members remaining the same at each meeting.

Data analysis

Before conducting further statistical analysis, the raw scores of the problem-solving skills, which ranged from 0 to 64, were transformed to scores ranging from 0 to 100 for easy interpretation by multiplying the raw scores by 100/64. Since the sample sizes in the individual and collaborative groups were small, the satisfaction of the normality assumption was tested to determine which statistical method – parametric or non-parametric statistics – would be better for analysis of the transformed problem-solving skills score data from the two groups. Given that the sample size in both groups was less than 30, the Shapiro-Wilk test was performed within the assumption of normality. The Shapiro-Wilk test demonstrated that neither the data of the student problem-solving skills scores in the individual group (W = 0.852, p = .002), nor the data of the student problem-solving skills scores in the collaborative group (W = 0.908, p = .024) were normally distributed. These results were supported by measures of skewness and kurtosis, which indicated that the distribution of the problem-solving skills scores in the individual group (skewness = −1.43, SE = 0.456, indicating that the distribution was left-skewed; kurtosis = 1.86, SE = 0.887, indicating that the distribution was heavy-tailed relative to the normal distribution) and the collaborative group (skewness = −0.843, SE = 0.456, indicating that the distribution was left-skewed; kurtosis = −1.86, SE = 0.887, indicating that the distribution was light-tailed relative to the normal distribution) departed from the normal distribution (Kim, 2013) (see Figure 3). Therefore, non-parametric statistics were used to analyse the data on students' problem-solving skills scores through descriptive and inferential statistics. Inferential statistics were used to test the hypotheses proposed in the study.

Figure 3

Histograms of the problem-solving scores in the individual and collaborative groups



Given the suitability of non-parametric statistics for inferential analysis based on the distribution of the data in each group, the hypotheses were tested using the one-sample rank test via Wilcoxon rank and the Mann-Whitney U test. Both of these tests, as well as the descriptive analysis, were performed using jamovi 2.3.21 (R Core Team, 2021; The jamovi project, 2022). The Wilcoxon rank was used to test the hypothesis regarding the effectiveness of treatment in each group in promoting problem-solving skills by setting 75 as the test value, while the Mann-Whitney *U* test was used to examine whether there was a difference in the effectiveness of the treatment received by each group on problem-solving skills. It was found that both treatments were effective. A significance level (α) of 0.05 was used to determine statistical significance for testing both hypotheses, where the resulting *p*-value corresponding to a test statistic that is greater than or equal to the significance level indicates that there is insufficient evidence to reject the null hypothesis proposed in the hypothesis testing. The results regarding the differences in effectiveness between the two treatments were then used to investigate whether one group performed better in terms of problem-solving skills than the other group. The raw score data and the results of the descriptive and inferential statistics of the study format are available in jamovi file at Open Science Framework (OSF): https://osf.io/xf6v8/.

Results

The present study focused on examining the potential of presenting erroneous examples to students in individual and collaborative settings in promoting their problem-solving skills. In order to achieve this objective, we involved students from two classes, with each class receiving different treatment in terms of the learning settings applied when facilitating learning about the application of the first derivative. Descriptive statistics revealed that the median of the students' problem-solving skills scores from the individual group (*Mdn* = 90.6, range = 42.2) was higher than the median of those from the collaborative group (*Mdn* = 88.3, range = 51.6). Table 1 shows detailed non-parametric descriptive statistics from the data on the problem-solving skills scores of the students from the two groups in the study, while Figure 4 presents descriptive statistics visually in the form of a violin plot.

Table 1

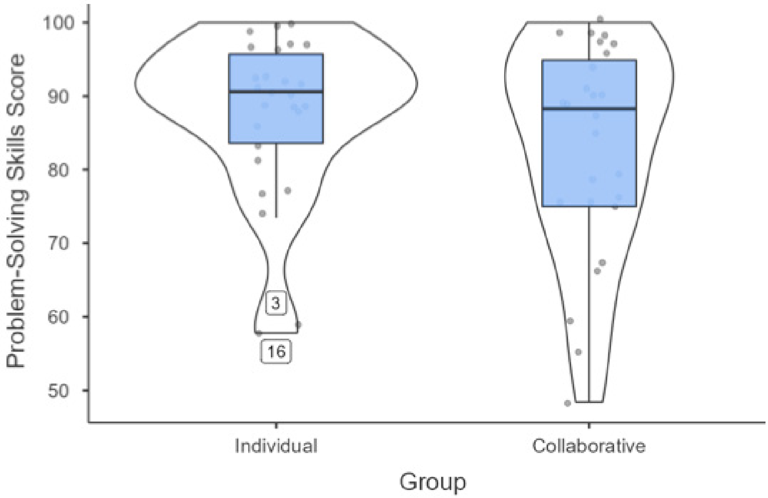
Non-parametric descriptive statistics of the students' problem-solving skills scores

Group	*N*	*M*	*SD*	*Min.*	*Max.*	*Q1*	*Q2*	*Q3*	*IQR*
Individual	26	87.5	11.1	57.8	100	83.6	90.6	95.7	12.1
Collaborative	26	82.9	14.6	48.4	100	75.0	88.3	94.9	19.9

Note. Min. = The minimum score a student achieved on the test, Max. = The maximum score a student achieved on the test, *Q*1 = The first quartile or the 25th percentile, *Q*2 = The second quartile (median) or the 50th percentile, *Q*3 = The third quartile or the 75th percentile, and *IQR* = Interquartile range (*Q*3 – *Q*1).

Figure 4

Violin plots of the problem-solving skills scores of the students in the individual and collaborative groups



Based on the Wilcoxon rank test, there is evidence to confirm that engaging students in learning the application of the first derivative using erroneous examples in an individual setting (*Mdn* = 90.6) was significantly able to promote students' problem-solving skills, $W = 321$, $p < 0.001$. In addition, the rank biserial correlation showed the existence of a large effect size (r_{rb} = 0.829). The Wilcoxon rank test also revealed evidence of the effectiveness of engaging students in learning in a collaborative setting (*Mdn* = 88.3) as well as the effectiveness of working on erroneous examples in promoting students' problem-solving skills, $W = 219$, $p = 0.015$. In addition, the rank biserial correlation demonstrated a small effect size (r_{rb} = 0.245). This finding confirms what was hypothesised in the study.

In conducting the Mann-Whitney *U* test on the data on the problem-solving skill scores ranging from 0 to 100 in both groups, no evidence was found to confirm a significant difference ($U = 281$, $p = 0.300$) between the students from the individual group (*Mdn* = 90.6) and those from the collaborative group (*Mdn* = 88.3) in terms of their problem-solving skills, with rank biserial correlation demonstrating a trivial effect size (r_{rb} = 0.169). This suggests that there is not sufficient statistical evidence to say that the implementation of a particular learning setting will provide more benefits than other learning settings when learning activities are facilitated by presenting erroneous examples to students.

Thus, the hypothesis put forward in the study that engaging students to detect, explain and fix errors through discussion or collaborative work is better than through individual work was not confirmed by the empirical data of the study.

Discussion and Conclusion

The present study aimed to uncover the effects of engaging students in working with erroneous examples in individual and collaborative settings on their problem-solving skills, particularly when solving first derivative of a function application problems. The study found that the use of erroneous examples in an individual setting promotes students' problem-solving skills, thus confirming one of the study's hypotheses. This finding is in accordance with the results of a number of studies (e.g., Adams et al., 2012; Barbieri & Booth, 2016; Huang, 2017; McLaren et al., 2012, 2015) demonstrating that facilitating students to find, explain and fix errors in erroneous examples can promote conceptual and procedural understanding. Both types of understanding have a crucial role in students' success in solving problems and in developing their problem-solving skills.

Analysis of the data collected in the present study suggests that effectively engaging students in a collaborative setting to detect, explain and correct errors and solve problems similar to problems in erroneous examples can facilitate their problem-solving skills, which supports the finding of a study conducted by Yang et al. (2016). In a collaborative setting, students were facilitated to discuss and help each other in carrying out activities such as detecting errors in problem solving, explaining the existing errors (why they can be considered errors) and correcting errors. Such activities help students to develop conceptual understanding and logical reasoning, as well as the ability to formulate, represent and solve problems (Burkhardt & Swan, 2017). Furthermore, Borasi (Zhao & Acosta-Tello, 2016) suggests that discussing errors in erroneous examples allows students to improve their ability to think critically about mathematical concepts, to think about and try various problem-solving strategies, and to engage in self-reflection, all of which fundamentally support students' success in solving problems. Although the use of erroneous examples proved to be effective in supporting the development of students' problem-solving skills in the present study, the results do not provide evidence to suggest that one learning setting is better than the other. The study thus provides support for the potential of engaging students to read mathematical texts in developing problem-solving skills (Papadopoulos & Kyriakopoulou, 2022). Regardless of the type of learning setting used, this potential is extended when the activity of reading mathematical texts in the form of step-by-step problem solving is coupled with detecting, explaining and fixing errors contained in a step-by-step solution, which also has great potential for the development of students' problem-solving skills.

Regarding the effectiveness of individual and collaborative settings in example-based learning, previous studies have also demonstrated that the application of a collaborative setting is no better than an individual setting in mathematics learning (e.g., Retnowati et al., 2010, 2016), although a study conducted by Rushton (2018) suggested that when students were allowed to learn mathematics through error analysis activities collaboratively in pairs or small groups, their learning achievements showed promising improvements. Furthermore, a study conducted by Yang et al. (2016) found that the use of erroneous examples in learning on the topic of decimals in a collaborative setting was better than in an individual setting in terms of the students' transfer of procedural knowledge. In the present study, however, it was found that presenting erroneous examples in a collaborative setting was no more effective than in an individual setting with regard to students' problem-solving skills. A possible reason for this is that, in a group setting, some students relied more on other group members who were considered more capable in dealing with erroneous examples. This would prevent such students from obtaining maximum benefits offered by the collaborative setting in learning applications of the first derivative through erroneous examples.

The present study seeks to contribute to a topic that is still under researched, namely the effect of learning settings in the use of erroneous examples. Students from two classes received different treatment in terms of learning settings, but were all equally facilitated to learn mathematics through activities involving finding errors, explaining the detected errors and correcting them, as well as solving problems similar to those in the erroneous examples. The results of the study suggest that mathematics learning carried out by presenting erroneous examples and having students work on those erroneous examples is effective for promoting students' problem-solving skills, both when learning is designed in individual and collaborative settings. The study also indicated that there were no significant differences between the two settings. These findings mean that, in practice, teachers can create learning that provides greater opportunities for students to learn from errors, and that their problem-solving skills can develop through varied learning settings: sometimes using an individual setting and sometimes using a collaborative setting to create differentiated instruction.

The study does, however, have certain limitations. Firstly, the sample size in both groups was small, i.e., less than 30. Future studies should use a larger sample size in each group. A larger sample size allows us to obtain data that are normally distributed or close to normal distribution, so that analysis can be carried out using parametric statistics, which are believed to be more robust than non-parametric statistics. The problem-solving skills scores data in this study, based on statistical test and graphical approaches, shows a distribution that deviates from a normal distribution, which is why non-parametric statistics were used in analysing the data. This is another limitation of the present study. Given the small sample sizes in each group, we did not further explore the effects of existing treatments based on student gender, although this factor might suggest interesting findings regarding the use of erroneous examples in mathematics learning in individual and collaborative settings. Future studies are therefore expected to consider this aspect.

In addition, the current study used two experimental groups without using a control group. The existing literature suggests that the use of a control group in a quasi-experimental study is generally advisable, although in some circumstances it is not always possible to include such a group, as the control group can be considered the best baseline to the treatment condition (Rogers & Révész, 2020). A control group was not included in the present study, as the aim was not to demonstrate the superiority of the two treatments over conventional learning methods, but rather to promote the use of erroneous examples in mathematics learning as an alternative to the conventional learning method. However, the absence of a control group in the study resulted in insufficient information regarding the position of the two treatments relative to a conventional learning method. The provision of such information would provide more insight into opportunities to engage students to detect, explain and fix errors through the presentation of erroneous examples. Future studies are therefore advised to use control groups to address this limitation. Lastly, inspired by studies conducted by Khasawneh et al. (2022, 2023) and Rushton (2018), which explored the effects of the use of erroneous examples based on quantitative and qualitative approaches, we note that the present study did not employ a qualitative approach. We therefore suggest that future studies explore students' opinions, perceptions, judgments or feedback on their involvement in learning activities facilitated through the presentation of erroneous examples in individual and collaborative settings.

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